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INVARIANT IMBEDDING AND THE INTEGRATION OF HAMILTON'S EQUATIONS

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PREFACE

Part of RAND's long-range research program in mathematical physics has been devoted to obtaining improved computational techniques for determining critical mass (a contribution to neutron transport theory), and to developing new analytical approaches to transport theory, in general. This work has led to new analytical techniques for the integration of the basic equations of classical particle dynamics, providing another example of how basic research in one area frequently sheds light on a seemingly unrelated one.

This Memorandum deals with theoretical problems and basic equations in analytical dynamics. It should be of interest to those concerned with orbit determination, trajectory optimization, and related fields.

SUMMARY

Based on earlier work on the application of the principle of invariant imbedding to problems in transport theory, the authors describe a countercurrent flow model which can be associated with any mechanical process described by Hamilton's equations of motion. Then they show how the determination of certain reflection and transmission functions, which occur naturally in the transport process, leads to the solution of Hamilton's equations. Some examples are given, and some possible future applications are sketched.

CONTENTS

I. INTRODUCTION

Much of classical analytical dynamics is concerned with the integration of Hamilton's canonical equations of motion [1,2]. The primary general approach is through the Hamilton-Jacobi theory [2,3], which involves determining a complete solution of the Hamilton-Jacobi equation, a nonlinear partial differential equation of first order. Our aim in this Memorandum is to present an alternative approach which has several conceptual and manipulative advantages:

1. Only functions with clear physical meanings are introduced (no "action functions" are used).

2. The functions are solutions of quasilinear, as opposed to nonlinear, first order partial differential equations.

3. The functions are solutions of initial value problems which facilitates their computational solution.

4. The methodology is equally valid for equations of motion which do not arise from variational principles.

In earlier work we discussed the application of principles of invariance to the analysis of transport processes [4,5]. There it was quite natural to introduce certain transmission and reflection functions and to derive equations for them [6,7]. In this Memorandum we associate a countercurrent flow model with each dynamical process [8], and we interpret the reflection and transmission functions in mechanical terms. We show that the solution of the canonical equations of motion can be given in terms of these reflection and transmission functions. Some examples of the methodology are given involving harmonic oscillations and motion in uniform and central inverse-square fields of force. Some possible future uses are sketched.

Other uses of functional equation techniques in mechanics are given in [9], [10] and [11].

II. HAMILTON'S EQUATIONS AND THE COUNTERCURRENT FLOW MODEL OF MECHANICAL PROCESSES

Let us consider a mechanical system with N degrees of freedom with a hamiltonian $H(q,p,t)$, where q and p are N-dimensional vectors. The canonical equations of motion are

(1) $$\dot{q}_i = H_{p_i}$$

(2) $$-\dot{p}_i = H_{q_i} ,$$

equations which are to hold for

(3) $$0 \le t \le T$$

and

(4) $$i=1,2,\ldots,N .$$

Similar equations occurred in [6] and [7]. With these equations we associate a steady flow in a tube of length T. The i^{th} component of the vector $q(t)$, $q_i(t)$, represents the amount of material of the i^{th} type passing the point t per unit of time and moving toward the right. Similarly, the i^{th} component of the vector $p(t)$, $p_i(t)$ represents the amount of material of the i^{th} type passing the point t per unit of time and moving toward the left. Furthermore, we wish to assume that the various flows interact with one another and with the medium. The exact nature of the interaction in a section of the rod extending from t to $t + \Delta$ is assumed given by the relations

(5) $$q_i(t + \Delta) = q_i(t) + \Delta H_{p_i}(q(t), p(t), t) + o(\Delta)$$

and

$$p_i(t) = p_i(t + \Delta) + \Delta H_{q_i}(q(t), p(t), t) + o(\Delta) , \qquad (6)$$

$$i=1,2,\ldots,N .$$

Equations (1) and (2) represent limiting forms of these equations obtained by letting Δ tend to zero. Finally, let us suppose that there is a steady-state input flow to the right at $t = 0$ given by

$$q_i(0) = w_i , \qquad i=1,2,\ldots,N \qquad (7)$$

and a steady-state input flow to the left at $t = T$ given by

$$p_i(T) = y_i , \qquad i=1,2,\ldots,N . \qquad (8)$$

The countercurrent flow model of a mechanical process is shown diagramatically in Fig. 1.

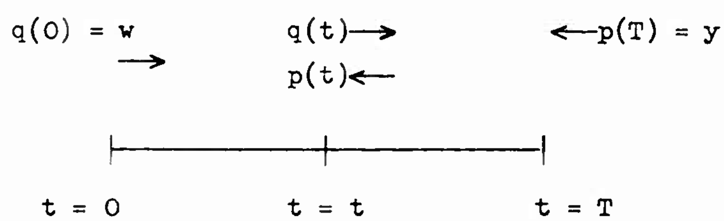

Fig. 1 -- The Countercurrent Flow Model of a Mechanical Process

III. THE REFLECTION VECTOR OR THE TERMINAL DISPLACEMENT VECTOR

Some questions now arise in connection with the physical situation which has just been sketched:

1. What are the internal flows that are produced as a result of the impressed flows at the ends of the rod? In terms of the original mechanical process, if the initial displacements and terminal moments are given, what are the intervening displacements and moments?
2. What is the flow to the right at the right end of the tube and the unknown flow to the left at the left end of the rod? In terms of the original mechanical process, what are the unknown moments at the initial instant and the unknown displacements at the termination of the process?

We shall first focus our attention on question 2 and then show how the answer to that question leads us to the answer of question 1.

The unknown flows to the right at the right end of the tube depend upon the two input flows w and y, and also upon the length of the tube, T. Let us then introduce the functions

(1) $r(i,w,y,T)$ = the amount of material of type i emerging per unit time from the right end of the tube extending from 0 to T, due to an input flow w at the left end of the tube and an input flow y at the right end of the tube, $i = 1,2,\ldots,N$.

We shall derive a system of partial differential equations for these functions by relating the functions $r(i,w,y,T + \Delta)$ to the functions

$r(i,w,y,T)$. The first set of functions refers to a tube extending from 0 to $T + \Delta$ and the second to one from 0 to T.

To terms involving the first power in Δ, the desired relations are

$$(2) \qquad r(i,w,y,T+\Delta) = H_{p_i}(r,y,T)\Delta + \Delta \sum_{m=1}^{N} \frac{\partial r(i,w,y,T)}{\partial y_m} H_{q_m}(r,y,T) + r(i,w,y,T) ,$$

$$i=1,2,\ldots,N .$$

Fig. 2 The Physical Situation

The first term on the right hand side of Eq. (2) arises from the interaction of the fluxes y and $r(i,w,y,T)$ in the interval T to $T + \Delta$. The second arises from the reflection of particles from the tube extending from 0 to T due to particles incident at $t = T$, modified by a passage through the section of tube extending from T to $T + \Delta$.

Upon letting Δ tend to zero Eq. (2) becomes

$$(3) \qquad \frac{\partial r(i,w,y,T)}{\partial T} = H_{p_i}(r,y,T) + \sum_{m=1}^{N} H_{q_m}(r,y,T) \frac{\partial r(i,w,y,T)}{\partial y_m} ,$$

$$i=1,2,\ldots,N .$$

This is the desired quasilinear system of partial differential equations for the unknown displacements in terms of the initial displacement vector w, the duration of the process T and the terminal momentum vector y. This is merely a reinterpretation of equations given in [6] and [7] for the flow problems.

IV. THE TRANSMISSION VECTOR OR THE INITIAL MOMENTUM VECTOR

Let us introduce the transmission functions

(1) $\tau(i,w,y,T)$ = the amount of material of type i emerging from the left end of the tube extending from 0 to T due to an input flow w at the left end of the tube and an input flow of y at the right end of the tube, $i = 1,2,\ldots,N$

Then, as is indicated in [6] and proved in [7], the functions $\tau(i,w,y,T)$ satisfy the system of equations

(2) $$\frac{\partial\tau(i,w,y,T)}{\partial T} = \sum_{m=1}^{N} H_{q_m}(r,y,T)\,\frac{\partial\tau(i,w,y,T)}{\partial y_m} .$$

Notice that the Eqs. (3.3) and (2) form a system of quasilinear first-order partial differential equations with the same principle parts. These are discussed in [3, pp. 139-145].

V. INTEGRATION OF HAMILTON'S EQUATIONS USING THE r AND τ FUNCTIONS

Let us begin by noting that the reflection and transmission vectors are easily determined for tubes of zero length. We have

(1) $$r(i,w,y,0) = w_i$$

and

(2) $$\tau(i,w,y,0) = y_i \,, \qquad i=1,2,\ldots,N .$$

Suppose next that we can solve the systems of quasilinear partial differential Eqs. (3.3) and (4.2), subject to the conditions given above. Then we see that we obtain the integrals of the canonical Eqs. of motion (2.1) and (2.2) in the form

(3) $$q_i(t) = r(i,w,p(t),t)$$

(4) $$b_i = \tau(i,w,p(t),t) \,, \qquad i=1,2,\ldots,N .$$

There are 2N arbitrary constants, w_i and b_i, $i = 1,2,\ldots,N$. The constants b_i represent the flows to the left at $t = 0$ due to input flows of w and $p(t)$ at $t = 0$ and $t = t$, respectively, to a tube extending from 0 to t.

It is easy to verify directly that if $r(i,w,y,T)$ and $\tau(i,w,y,T)$ satisfy the quasilinear partial differential Eqs. (3.3) and (4.2) respectively, and if the functions $q_i(t)$ and $p_i(t)$ satisfy Eqs. (3) and (4), then the functions $q_i(t)$ and $p_i(t)$ are solutions of Hamilton's equations.

First differentiate the equations in (4) with respect to t to obtain

$$(5) \qquad 0 = \sum_{j=1}^{N} \frac{\partial \tau(i,w,p,t)}{\partial p_j} \frac{dp_j}{dt} + \frac{\partial \tau(i,w,p,t)}{\partial t} \, ,$$

$$i=1,2,\ldots,N \, .$$

If now we view the equations in (5) as a set of linear algebraic equations for the unknowns dp_j/dt, and if we assume that

$$(6) \qquad \det\left(\frac{\partial \tau(i,w,p,t)}{\partial p_j}\right) \neq 0 \, ,$$

then on comparing Eqs. (5) and (4.2) we see that

$$(7) \qquad - \frac{dp_j}{dt} = \frac{\partial H}{\partial q_j} \, .$$

In addition, if we differentiate Eq. (3) with respect to t and use Eqs. (7) and (3.3), we obtain the relations

$$(8) \qquad \frac{dq_i}{dt} = \sum_{j=1}^{N} \frac{\partial r(i,w,p,t)}{\partial p_j} \frac{dp_j}{dt} + \frac{\partial r(i,w,p,t)}{\partial t}$$

$$= \sum_{j=1}^{N} \frac{\partial r(i,w,p,t)}{\partial p_j} \left(- \frac{\partial H}{\partial q_j}\right) + \frac{\partial r(i,w,p,t)}{\partial t}$$

$$= H_{p_i} \, .$$

But Eqs. (7) and (8) are the desired canonical equations.

VI. EXAMPLES

We now turn to some illustrations. Let us first consider the harmonic oscillator for which

$$(1) \qquad H(q,p,t) = \frac{p^2}{2m} + \frac{1}{2} k q^2 .$$

The canonical equations of motion are

$$(2) \qquad \dot{q} = p/m , \qquad q(0) = w ,$$

and

$$(3) \qquad -\dot{p} = k q , \qquad p(T) = y ,$$

where q and p are now scalar quantities. Equation (3.3) becomes

$$(4) \qquad \frac{\partial r}{\partial T} = \frac{y}{m} + k r \frac{\partial r}{\partial y}$$

with

$$(5) \qquad r(w,y,0) = w .$$

Equation (4.2) becomes

$$(6) \qquad \frac{\partial \tau}{\partial T} = k r \frac{\partial \tau}{\partial y}$$

with

$$(7) \qquad \tau(w,y,0) = y .$$

For simplicity consider those processes for which

$$(8) \qquad w = 0 ,$$

i.e., processes for which the initial displacement is zero. Then the linearity of Eqs. (2) and (3) suggests writing the function r in the form

$$r = R(T)y \;. \tag{9}$$

Substitution of this in Eq. (4) yields an ordinary differential equation of Riccati type for the function R(T)

$$\frac{dR}{dT} = \frac{1}{m} + k\,R^2 \,, \qquad R(0) = w = 0 \;. \tag{10}$$

The solution of this equation is

$$R(T) = \frac{1}{\sqrt{km}} \tan \sqrt{\frac{k}{m}}\; T \,, \tag{11}$$

so that

$$r = \frac{y}{\sqrt{km}} \tan \sqrt{\frac{k}{m}}\; T \;. \tag{12}$$

In addition, the solution of Eqs. (6) and (7) is

$$\tau = y \sec \sqrt{\frac{k}{m}}\; T \;. \tag{13}$$

Consequently the theorem of Section V leads to the integrals

$$g(t) = \frac{p(t)}{\sqrt{km}} \tan \sqrt{\frac{k}{m}}\; t \tag{14}$$

and

$$b = p(t) \sec \sqrt{\frac{k}{m}}\; t \,, \tag{15}$$

from which we obtain the standard results

$$p(t) = b \cos \sqrt{\frac{k}{m}}\; t \tag{16}$$

and

$$q(t) = \frac{b}{\sqrt{km}} \sin \sqrt{\frac{k}{m}}\; t \;. \tag{17}$$

Next let us consider motion of a particle of mass m in a uniform field of force. We choose rectangular coordinates $x = q_1$ and $y = q_2$ so that the potential energy is given by the formula

$$U(x,y) = m\,g\,y \tag{18}$$

We denote the x and y components of momenta by p_x and p_y. Then the hamiltonian is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + m\,g\,y\,, \tag{19}$$

and the canonical equations of motion are

$$\dot{x} = p_x/m\,, \qquad -\dot{p}_x = 0\,, \tag{20}$$

$$\dot{y} = p_y/m\,, \qquad -\dot{p}_y = m\,g\,. \tag{21}$$

The partial differential equations for the reflection functions become

$$\frac{\partial r_1}{\partial T} = \frac{y_1}{m} + 0 \cdot \frac{\partial r_1}{\partial y_1} + m\,g\,\frac{\partial r_1}{\partial y_2} \tag{22}$$

and

$$\frac{\partial r_2}{\partial T} = \frac{y_2}{m} + 0 \cdot \frac{\partial r_2}{\partial y_1} + m\,g\,\frac{\partial r_2}{\partial y_2}\,. \tag{23}$$

The solutions of these equations, subject to the conditions (5.1), are

$$r_1 = \frac{y_1}{m}\,T + w \tag{24}$$

$$r_2 = \frac{y_2}{m}\,T + \frac{1}{2}\,g\,T^2 + w_2\,. \tag{25}$$

The equations for the transmission functions are

$$\frac{\partial \tau_1}{\partial T} = m\,g\,\frac{\partial \tau_1}{\partial y_2} \tag{26}$$

and

$$\frac{\partial \tau_2}{\partial T} = m\,g\,\frac{\partial \tau_2}{\partial y_2}\ . \tag{27}$$

The solutions, subject to the conditions (5.2), are

$$\tau_1 = y_1 \tag{28}$$

$$\tau_2 = y_2 + m\,g\,T\ . \tag{29}$$

Then Eqs. (24) and (25) lead to the integrals

$$x(t) = \frac{p_x(t)}{m}\,t + w_1 \tag{30}$$

$$y(t) = \frac{p_y(t)}{m}\,t + \frac{1}{2}\,g\,t^2 + w_2\ , \tag{31}$$

and Eqs. (28) and (29) lead to the integrals

$$b_1 = p_x(t) \tag{32}$$

$$b_2 = p_y(t) + m\,g\,t\ . \tag{33}$$

But Eqs. (30) - (33) are a form of the solution of the canonical Eqs. (20) and (21), where b_1 and b_2 are the initial momenta and w_1 and w_2 are the initial displacements.

Finally, to show the effect of kinosthenic coordinates, we derive the law of the constancy of angular momentum for the

motion of a particle in an inverse-square central-force field. Using polar coordinates and an obvious notation, we see that the hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + r^{-2}\,p_\theta^2\right) - \frac{G}{r}\,. \tag{34}$$

Hamilton's equations become

$$\dot{r} = \frac{p_r}{m}\,, \qquad -\dot{p}_r = \frac{G}{r^2} - \frac{p_\theta^2}{mr^3}\,, \tag{35}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}\,, \qquad -\dot{p}_\theta = 0\,. \tag{36}$$

Upon letting the subscript "1" refer to the radial components and the subscript "2" refer to the angular components, we find that the partial differential equations for the reflection and transmission functions are

$$\frac{\partial r_1}{\partial T} = \frac{y_1}{m} + \left(\frac{G}{r_1^2} - \frac{y_2^2}{m\,r_1^3}\right)\frac{\partial r_1}{\partial y_1} \tag{37}$$

$$\frac{\partial r_2}{\partial T} = \frac{y_2}{m\,r_1^2} + \left(\frac{G}{r_1^2} - \frac{y_2^2}{m\,r_1^3}\right)\frac{\partial r_2}{\partial y_1} \tag{38}$$

$$\frac{\partial \tau_1}{\partial T} = \left(\frac{G}{r_1^2} - \frac{y_2^2}{m\,r_1^3}\right)\frac{\partial \tau_1}{\partial y_1} \tag{39}$$

$$\frac{\partial \tau_2}{\partial T} = \left(\frac{G}{r_1^2} - \frac{y_2^2}{m\,r_1^3}\right)\frac{\partial \tau_2}{\partial y_1}\,. \tag{40}$$

Equation (40), coupled with the condition of Eq. (5.2) has the solution

(41) $$\tau_2 = y_2$$

Our integration principle tells us that during the process

(42) $$p_\theta = \text{const.} ,$$

which is the desired result.

VII. DISCUSSION

The advantages in providing some alternate mathematical formulations of physical processes are many. In the first place, some formulations are analytically more tractable than others for particular processes, and often the successful numerical solution is wholly dependent upon choosing the appropriate formulation.

Perhaps even more important are the conceptual advantages. If we can visualize various functions as reflected or transmitted fluxes, or velocities, or moments, we intuitively gain a stronger hold on the physical process and on the equations describing the process.

Too much of previous analysis has been inflexible, wedded to a particular analytical formulation. In forthcoming Memoranda we shall discuss some other classes of problems in versatile terms.

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